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NOTE

A Note on Convergence of Linear Positive Operators

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Recently Shisha and Mond [5, 6] and Devore [1] determined a quantitative estimate for the degree of convergence of linear positive operators to a given continuous function on a closed and bounded interval from the degrees of convergence to the test functions x^k, $k = 0, 1, 2$. Ditzian [2] modified these results to operators defined for functions on $[0, \infty)$ or $(-\infty, \infty)$.

Following Ditzian [2], we define operators of the type $\mathscr{H}(T, S, \mu)$.

DEFINITION 1.1. Let $T \subset (-\infty, \infty)$ be closed, let $-\infty < a < b < \infty$, and set $S = T \cap [a, b]$. Let $\mu(t)$ be a real-valued function on T satisfying $\mu(t) \geqslant 1$, $t \in T$. A sequence $\{L_n\}$ of linear positive operators is said to be of type $\mathscr{H}(T, S, \mu)$ if the domain of each L_n consists of all functions (or all measurable functions) f on T satisfying there

$$|f(t)| \leqslant M(f)(t^2 + 1)\,\mu(t)$$

and if

$$\|(L_n t^k)(x) - x^k\|_{C(S)} = O(1), \qquad n \to \infty, k = 0, 1, 2.$$

and

$$\|(L_n(t-x)^2\,\mu(t))(x)\|_{C(S)} \leqslant K\,\|L_n(t-x)^2(x)\|_{C(S)} = O(1),$$

K being a constant.

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The purpose of this note is to sharpen Theorem 2.1(C) of [2]. We prove

THEOREM 2.1. *Let A be a positive number and let $S_1 = [a_1, b_1] \subset [a, b]$ so that for some $\eta > 0$, $[a_1 - \eta, b_1 + \eta] \cap T \cap \{(-\infty, \infty) - [a, b]\} = \varnothing$. Let $\{L_n\}$ be a sequence of linear positive operators of type $\mathscr{H}(T, S, \mu)$ and let $f \in C^1[a, b]$. Then, for $n \geqslant 1$,*

$$\|L_n f - f\|_{C(S_1)} \leqslant \|L_n 1 - 1\| \, \|f\| + \|L_n(t-x)(x)\| \, \|f'\| + L\mu_n^2 + \mu_n \left[\|L_n 1\|^{1/2} + \frac{1}{2A} \right] \omega(f'; A\mu_n), \quad (2.1)$$

where ω is the modulus of continuity on $[a, b]$, and $\mu_n = \|L_n(t-x)^2(x)\|^{1/2}$, $\| \ \|$ being the sup-norm on S_1 and L a constant.

If $(L_n 1)(x) \equiv 1$ and $(L_n t)(x) \equiv x$, then (2.1) reduces to

$$\|L_n f - f\|_{C(S_1)} \leqslant \left(1 + \frac{1}{2A}\right) \mu_n \omega(f'; A\mu_n) + L\mu_n^2. \quad (2.2)$$

Proof. For $x \in S_1$, $t \in [a_1 - \eta, b_1 + \eta] \cap T$ we write

$$f(t) - f(x) = (t-x) f'(x) + \int_x^t (f'(\xi) - f'(x)) \, d\xi.$$

Using the proof in [7] and the inequality

$$|f'(\xi) - f'(x)| \leqslant \left(1 + \frac{|\xi - x|}{\delta}\right) \omega(f'; \delta), \qquad \delta > 0,$$

we get

$$\begin{aligned} &|(L_n f)(x) - f(x)(L_n 1)(x)| \\ &\quad \leqslant |L_n(t-x)(x)| \, |f'(x)| + \omega(f'; \delta) \, L_n \left[\left| \int_x^t \left(1 + \frac{|\xi - x|}{\delta}\right) d\xi \right| \right](x), \\ &\quad \leqslant |L_n(t-x)(x)| \, |f'(x)| + \omega(f'; \delta) \left[L_n|t-x|(x) + \frac{(L_n(t-x)^2)(x)}{2\delta} \right]. \end{aligned}$$

Choosing $\delta = A\mu_n$, we pursue a slight modification of the proof given by Ditzian [2], details of which we may omit.

EXAMPLE. The positive linear operators obtained from the inversion of the Weirstrass transform for measurable functions f on $(-\infty, \infty)$ are given by

$$(L_n f)(x) = \left(\frac{n}{4\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left(-(t-x)^2 \frac{n}{4}\right) f(t)\, dt, \qquad n \geqslant 1.$$

From [2] we have

$$(L_n 1)(x) = 1, \qquad (L_n t)(x) = x, \qquad (L_n t^2)(x) = x^2 + \frac{2}{n},$$

so that $(L_n(t-x)^2)(x) = 2/n$. Also $L_n \in \mathscr{K}(T, \mu)$, where $T = (-\infty, \infty)$ and $\mu(t) = e^{t^2/4}$. Choosing $A = 1/\sqrt{2}$ in (2.2), we get

$$\|L_n f - f\|_{C(S_1)} \leqslant \left(\frac{1+\sqrt{2}}{\sqrt{n}}\right) \omega\left(f'; \frac{1}{\sqrt{n}}\right) + L_1(a, b, \eta)\, n^{-1},$$

L_1 being a constant, which is sharper than the corresponding estimate due to Ditzian [2].

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It is worthwhile to point out that Theorem 4 of Mohapatra [3] and the result of Mond and Vasudevan [4] can be improved similarly.

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References

1. R. DEVORE, Optimal convergence of positive linear operators *in* "Proceedings of the Conference on Constructive Theory of Functions," pp. 101–119, Hungarian Academy of Sciences, Budapest, 1972.
2. Z. DITZIAN, Convergence of a sequence of positive linear operators, *J. Approx. Theory* **14** (1975), 296–301.
3. R. N. MOHAPATRA, Quantitative results on almost convergence of a sequence of positive linear operators, *J. Approx. Theory* **20** (1977), 239–250.
4. B. MOND AND R. VASUDEVAN, On approximation by positive linear operators, *J. Approx. Theory* **30** (1980, 334–336.
5. O. SHISHA AND B. MOND, The degree of convergence of Sequences of linear positive operators, *Proc. Nat. Acad. Sci. U.S.A.* **60** (1968), 1196–1200.
6. O. SHISHA AND B. MOND, The degree of approximation to periodic functions by linear positive operators, *J. Approx. Theory* **1** (1968), 335–339.
7. S. P. Singh, On the degree of approximation by Szâsz operators, *Bull. Austral. Math. Soc.* **24** (1981), 221–225.